Applied mathematics III

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1.1 Preliminary Concepts

- Differential equation (DE) is an equation containing the derivatives of 1(2) dependent variables with respect to 1(2) independent variables
- It can be classified respect to type, order and linearity. (i) DE can be classified into two types: ODE and PDE
- ODE is a DE containing the derivatives of 1(2) dependent variables wrt a single independent variable.
- PDE is a DE containing the derivatives of 1(2) dependent variables wrt 1(2) dependent variables.

  Example: $\frac{dy}{dx} + 2xy = 1$ (ODE), $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 1$ (PDE)

  (ii) DE can be classified into $1^{\text{st}}, 2^{\text{nd}}, \cdots, \text{n}^{\text{th}}$-order
- The order of DE is the highest derivative present in the equation

- (iii) An nth-order ODE

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \cdots, y^{n-1}) \tag{1}$$

is said to be linear if f is linear in $y, y', y'', \cdots, y^{n-1}$ and their coefficients depend at most on the independent variable x

- Otherwise it is nonlinear.

Example: Classify the following ODEs as type, order and linearity

1. $(1-x)y' + 2xy = e^x$ (Linear 1st-order ODE)
2. $y'' + y' + y = \sin x$ (Linear 2nd-order ODE)
3. $y'' + (1-y)y' + y^2 = 0$ (Non-linear 2nd-order ODE)
4. $$\left\{ y' = f(x, y),\ y(x_0) = y_0 \quad (1^{st}\text{IVP}) \right.$$
5. $$\left\{ y'' = f(x, y, y'),\ y(x_0) = y_0, y'(x_0) = y_1 \quad (2^{nd}\text{IVP}) \right.$$

...cont'

Existence and Uniqueness of Solutions

- A function $y(x) = \phi(x)$ is the solution of the n$^{\text{th}}$-order ODE (1) if $\dfrac{d^n\phi(x)}{dx^n} = f(x, \phi(x), \phi(x)', \cdots, \phi(x)^{n-1})$

Theorem

Consider the linear first order initial value problem:

$$y' + p(t)y = q(t),\ \ y(t_0) = y_0$$

If the functions p and q are continuous on an open interval (α, β) containing the point $t = t_0$, then there exists a unique solution $y = \phi(t)$ that satisfies the IVP for each t in (α, β).

Example: Verify $y = c/x$ (c -constant) is the solution of ODE $xy' + y = 0$
Ans. $xy' + y = 0 \Leftrightarrow x(-c/x^2) + c/x \equiv 0.$

Theorem

Consider the nonlinear first order initial value problem:

$$y' = f(t, y),\ y(t_0) = y_0$$

Suppose f *and* $\partial f/\partial y$ *are continuous on some open rectangle* $(t, y) \in (\alpha, \beta) \times (\gamma, \delta)$ *containing the point* (t_0, y_0). *Then in some interval* $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$, *then there exists a unique solution* $y = \phi(t)$ *that satisfies the IVP.*

1.2 Separable Differential Equations

- The 1^{st}-order $dy/dx = f(x, y)$ is separable equation if it can be written in the form $dy/dx = g(x)h(y)$ or $g(x)dx + h(y)dy = 0$
- Otherwise it is non-separable DE

  Method of solving separable DE- Apply separation of variable:

$$\int \frac{1}{h(y)} dy = \int g(x)dx + c \quad \text{(justify!)}$$

...cont'

Example: Solve the following DE if it is only separable equation

1. $yy' = x$. Ans. $y^2/2 = x^2/2 + c$
2. $y' - y = x$.
3. $y' - (x+1)y = -x - 1$. Ans. $y = 1 + ce^{x^2/2+x}$
4. $y' = \sin x \cos y$. (Exercise)

Some non-separable DEs can be reduced to separable form:

- A DE of the form $y' = f(y/x)$,(where $f(tx, ty) = t^n f(x, y)$ is homogeneous function of degree n) can be made separable by transformations that introduce for y a new unknown function u, i.e., set $y = ux$.
- To solve ODE $y' = f(y/x)$, let $y = ux$.
  Then $y' = xdu/dx + u \Leftrightarrow f(u) = xdu/dx + u$
  $\Leftrightarrow \dfrac{du}{dx} = \dfrac{1}{x}(f(u) - u)$ (Separable equation)
  Hence $\int \dfrac{1}{f(u) - u} du = \int \dfrac{1}{x} dx + c$ is the solution of $y' = f(y/x)$.

Example: Solve the following ODE

1. $xyy' = 4x^2 + y^2$. Ans. $y^2/2x^2 = 4\ln x + c$ is the solution
2. $xy' = x + y$. Ans. $y = x\ln x + cx$ is the solution
3. $yy' = x + y$. (Exercise)

1.3 Homogeneous Differential Equation

- A linear first-order ODE

$$y' + p(x)y = q(x) \tag{2}$$

(i) is homogeneous when $q(x) = 0$
(ii) is non-homogeneous when $q(x) \neq 0$

- NB. Any homogeneous ODE can be separable (Justify).

Example: Solve the following ODE if it is homogeneous

1. $xy' + (x-1)y = 0$. Ans. $y = cxe^{-x}$ is the solution
2. $xy' = x + y$. (Exercise)

1.4 Exact Differential Equations

- A first-order ODE

$$M(x,y)dx + N(x,y)dy = 0 \tag{3}$$

is exact if there exists a function $f(x,y)$ such that

$$df(x,y) = M(x,y)dx + N(x,y)dy = 0,$$

where df is the total differential of f

- Test for exactness: The necessary condition for exactness of Eq. (3) is $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$ (Justify!)
- Suppose Eq. (3) is exact. Hence there is $f(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0$$

$$\Leftrightarrow \frac{\partial f(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

...cont'

- Assuming the second-order mixed derivatives of $f(x,y)$ are continuous, then we obtain
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial M(x,y)}{\partial y} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$
- The ODE Eq. (3) is non-exact if $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$

  To solve the exact Eq. (3) we follow the following steps:
- From $\frac{\partial f(x,y)}{\partial x} = M(x,y)$ or $\frac{\partial f(x,y)}{\partial y} = N(x,y)$, we get implicit solution $f(x,y) = \int M(x,y)dx + k(y)$ or $f(x,y) = \int N(x,y)dy + l(x)$, respectively.
- To find $k(y)$ we differentiate $f(x,y)$ wrt y or to find $l(x)$ we differentiate $f(x,y)$ wrt x.
- Thus, the general solution of exact Eq. (3) is the form $f(x,y) = c$, where c is arbitrary constant

...cont'

Example: Test the exactness of DE and if it is exact, then solve

1. $ydx + (x + \frac{2}{y})dy = 0$. Ans. Since $\frac{\partial M(x,y)}{\partial y} = 1 = \frac{\partial N(x,y)}{\partial x}$, given DE w/c is exact and its general solution is $f(x,y) = xy + 2\ln y = c$
2. $\cos(x+y)dx + (3y^2 + xy + \cos(x+y))dy = 0$. Ans. Since $\frac{\partial M(x,y)}{\partial y} = -\sin(x+y) \neq y - \sin(x+y) = \frac{\partial N(x,y)}{\partial x}$, the given DE is non-exact DE
3. $xydx + (\frac{x^2}{2} + \frac{1}{y})dy = 0$. (Exercise)

1.5 Integrating Factors

- Consider the Eq. (3) is non-exact. Suppose that F is an integrating factor of Eq. (3) so that

$$FM(x,y)dx + FN(x,y)dy = 0 \quad (4)$$

- Now the DE we obtain from Eq. (4) is exact DE

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- Since Eq. (4) is exact, then $\dfrac{\partial(FM)}{\partial y} = \dfrac{\partial(FN)}{\partial x} \Leftrightarrow$

$$F_y M + F M_y = F_x N + F N_x \tag{5}$$

- Integrating factor F can be a function of x or a function of y:
- If $F = F(x)$ in Eq.(5), then $F_y = 0$ and $FM_y = F_x N + FN_x$

$$\Leftrightarrow \frac{1}{F} F_x = \frac{M_y - N_x}{N} \quad \text{(separable equation)}$$

- Hence $\dfrac{M_y - N_x}{N}$ is a function of x alone and

$F(x) = e^{\int \frac{M_y - N_x}{N} dx}$ is integrating factor of eq. (3).

- If $F = F(y)$ in Eq. (5), then $F_x = 0$ and $F_y M + F M_y = F N_x$

$$\Leftrightarrow \frac{1}{F} F_y = \frac{N_x - M_y}{M} \quad \text{(separable equation)}$$

- Hence $\dfrac{N_x - M_y}{M}$ is a function of y alone and $F(y) = e^{\int \frac{N_x - M_y}{M} dy}$ is integrating factor of eq. (3).
- NB.
  (i) if either $(M_y - N_x)/N$ or $(N_x - M_y)/M$ is constant, then the above formula still applied
  (ii) $F(x)Mdx + F(x)Ndy = 0$ or $F(y)Mdx + F(y)Ndy = 0$ is exact

Example: Find the integrating factor(s) of the following DE and solve it using integrating factor

1. $2ydx - xdy = 0$
2. $ydx - (x + 6y^2)dy = 0$ (Exercise)

- Ans. 1) $M = 2y$ and $N = -x$. The given DE is not exact because $M_y = 2 \neq -1 = N_x$
  To find integrating factor first we should compute:
- $\dfrac{M_y - N_x}{N} = -\dfrac{3}{x}$ which is a function of x alone and integrating factor $F(x) = \dfrac{1}{x^3}$ or
- $\dfrac{N_x - M_y}{M} = -\dfrac{3}{2y}$ which is a function of y alone and integrating factor $F(y) = \dfrac{1}{y^{3/2}}$
- Hence, we obtained two integrating factors:
  $F(x) = \dfrac{1}{x^3}$ and $F(y) = \dfrac{1}{y^{3/2}}$
- Now we can solve the given DE by using one of the two integrating factor
- Let us choose $F(x) = \dfrac{1}{x^3}$

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- Then multiplying the given DE by integrating factor $F(x) = \frac{1}{x^3}$ produces the exact DE:
$\frac{2y}{x^3}dx - \frac{1}{x^2}dy = 0$ and its solution is obtained as follows.
- Since the obtained DE is exact (i.e., $\frac{\partial(F(x)M)}{\partial y} = \frac{\partial(F(x)N)}{\partial x}$), there is $f(x, y)$:
$f(x, y) = \int \frac{2y}{x^3}dx + k(y) = -\frac{y}{x^2} + k(y)$
$\Leftrightarrow f_y = -\frac{1}{x^2} + k'(y) = -\frac{1}{x^2} = F(x)N$
$\Rightarrow k'(y) = 0 \Leftrightarrow k(y) = 0$
- Hence, $f(x, y) = -\frac{y}{x^2} = c$ is the general solution of the given DE
- Ans. 2) $\frac{N_x - M_y}{M} = -\frac{2}{y}$ and $F(y) = \frac{1}{y^2}$ is the only integrating factor and $f(x, y) = \frac{x}{y} - 6y = c$ is the general solution

1.6 Linear First-Order Differential Equations

- Consider the linear Eq. (2) which is not exact for $q(x) \neq 0$
- Hence, the general solution of Eq. (2) is given by

$$y(x) = e^{-\int p(x)dx} \int (q(x)e^{\int p(x)dx})dx + ce^{-\int p(x)dx} \qquad (6)$$

- To justify Eq. (6): Rewrite Eq. (2) as:

$$(p(x)y - q(x))dx + dy = 0$$

- We have $M = p(x)y - q(x),\ N = 1$ and
$F(x) = e^{\int \dfrac{M_y - N_x}{N} dx} = e^{\int p(x)dx}$ is integrating factor of Eq. (2)
- Now multiply Eq. (2) by $F(x) = e^{\int p(x)dx}$:

$$y' e^{\int p(x)dx} + p(x)ye^{\int p(x)dx} = q(x)e^{\int p(x)dx} \quad \text{(exact DE)}$$

$\Leftrightarrow d(ye^{\int p(x)dx}) = (q(x)e^{\int p(x)dx})dx$
$\Leftrightarrow ye^{\int p(x)dx} = \int (q(x)e^{\int p(x)dx})dx + c$
Hence, $y(x) = e^{-\int p(x)dx} \int \left(q(x)e^{\int p(x)dx}\right) dx + ce^{-\int p(x)dx}$

Example: Determine the linearity of the following DE and if so, solve

1. $y' + \frac{1}{x}y = x.$
2. $y' + y^2 = 1.$
3. $x^2y' + y = 1.$ (Exercise)

Ans. 1) The given DE is linear first-order, and $p(x) = \frac{1}{x},\ q(x) = x$

- Thus its general solution is given by formula Eq. (6) so that

$$y(x) = \frac{x^2}{3} + \frac{c}{x}$$

Ans. 2) The given DE is non-linear

Some non-linear first-order DE (e.g., Bernoulli equation) can be reduced to linear form:

- The Bernoulli equation is a non-linear DE given as:

$$y' + p(x)y = q(x)y^r, \tag{7}$$

  where r is any real number except 0 or 1.

- Aim: To transform Bernoulli equation (7) to linear
- First rewrite Eq. (7) as:
  $y^{-r}y' + p(x)y^{1-r} = q(x)$ and let $z = y^{1-r}$, then
  $z' = (1-r)y^{-r}y' \Leftrightarrow y' = \dfrac{1}{1-r}y^r z'$
- By substitution we obtain linear DE in z:

$$z' + (1-r)p(x)z = (1-r)q(x) \tag{8}$$

- To solve Bernoulli equation (7):
  (i) When $r = 0$ or $r = 1$, the Bernoulli equation (7) is linear and its solution is obtained by linear formula (6)
  (ii) When $r \neq 0$ or $r \neq 1$, the Bernoulli equation (7) is non-linear.
- To find its solution first we must change eq. (7) to eq. (8) form. Then the general solution is given as:

$$y^{1-r} = z = e^{-\int (1-r)p(x)dx} \left(\int \left((1-r)q(x)e^{\int (1-r)p(x)dx} \right) dx + c \right) \tag{9}$$

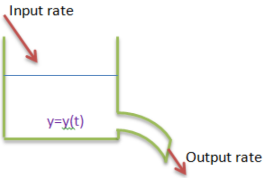
Example: solve the following DE

1. $y' + y = y^{2/3}$. Ans. $y^{1/3} = 1 + ce^{-x/3}$ is the general solution
2. $y' + \frac{1}{x}y = xy^4$. (Exercise)

1.7 Applications of First-Order ODEs

A) Mixing Problems

- Suppose that the mixing problem will involve a "tank" into which a certain substance will be added at a certain input rate and the substance will leave the system at a certain output rate (measured in the same units)
- Let $y = y(t)$ be the amount of the substance in the tank at time t



- Knowing the inflow rate and the concentration of the substance in the mixture flowing in allows us to determine the input rate at which the substance itself is entering the tank.

- The outflow concentration is simply the quantity of the substance present in the tank divided by the total volume of the mixture in the tank. This together with the output rate allows us to determine the output rate
- The rate at which the amount of substance changing in the tank over time t is given by

$$\begin{aligned}\frac{dy}{dt} &= \text{Input rate} - \text{Output rate} \\ &= (\text{conc. of infow})(r_{\text{in}}) - (\text{conc. of outflow})(r_{out}) \\ &= (\text{conc. of infow})(r_{\text{in}}) - \frac{(r_{\text{out}})y}{\text{volume of water} + (r_{\text{in}} - r_{\text{out}})t}\end{aligned} \tag{10}$$

- Here there are three possibilities: $r_{\text{in}} < r_{\text{out}}$, $r_{\text{in}} = r_{\text{out}}$ or $r_{\text{in}} > r_{\text{out}}$

Example: A large tank is filled to capacity with 500 gallons per pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank a rate of 5 gal/min. The well mixed solution is pumped out at a faster rate of 10 gal/min. Find the amount of salt in the tank at time t

Solution:

- Given: $r_{in} = 5$gal/min, $r_{out} = 10$gal/min, total volume of water=500 gal, conc. of inflow=2 pounds, conc. of outflow=$\dfrac{y}{500 + (r_{\text{in}} - r_{\text{out}})t}$ pounds
- Using eq. (10) the rate of amount of salt y at time t is given as:

$$\frac{dy}{dt} = 2 \times 5 - \frac{10y}{500 + (5 - 10)t} = 10 - \frac{2y}{100 - t}$$

$\Leftrightarrow$ (IVP):

$$\begin{cases} y' + \dfrac{2}{100 - t} y = 10 \text{ (linear first-order DE)} \\ y(0) = 0 \end{cases} \tag{11}$$

- To solve this linear DE we can apply formula (6) and so the amount of salt changing in the tank is $y(t) = 10 - \dfrac{(100-t)^2}{1000}$

(b) Newton's Law of Cooling

- The rate of change of the temperature of a body T is directly proportional to the difference between the temperature of a body T and temperature of the surrounding (room temp.) T_m, i.e.,

$$\frac{dT}{dt} = k(T - T_m) \quad \text{(linear first-order DE)} \tag{12}$$

where k is proportional constant

Example: 1) When a chicken is removed from an oven, its temperature is remeasured at 300°F. Three minutes later its temperature is 200°F. Find the temperature of chicken at room temperature 70°F.

2) An object y in a room whose temperature is kept at constant 60°c (the object) cools from 100°c to 90°c in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°c? Hint: $T(0) = 100, T(10) = 90, T(t) = 80, \;\; t = ?$ (Exercise)

Solution: 1) Given $T(0) = 300, T(3) = 200, T_m = 70$

- Use formula eq. (12) we obtain $dT/dt = k(T - 70)$ and apply formula eq. (6) we get $T(t) = 70 + ce^{kt}$
- Hence $T(t) = 70 + 230e^{\frac{1}{3}\ln(13/23)t}$ is the temperature of chicken

(c) Electric Series Circuits

- There are two series circuits:
  (i) LR-series circuit (contains only a resistor R and inductor L)
  (ii) RC-series circuit (contains only a resistor R and capacitor C)

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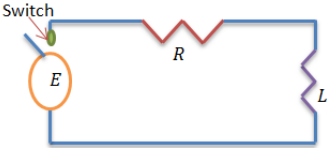
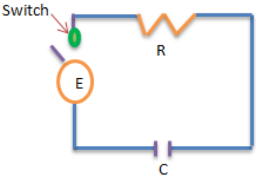
- Look at the following figures



(a)



(b)

- By Kirchhoff's second law the sum of the voltage drop across the inductor (LdI/dt) and the voltage drop across the resistor (IR) is the same as the impressed voltage ($E(t)$) on the circuit. That is,

$$L\frac{dI}{dt} + RI = E(t)$$

$$\Leftrightarrow \frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L} \quad \text{(linear first-order DE)}$$

...cont'

- By Kirchhoff's second law the sum of the voltage drop across the resistor (RdQ/dt) and the voltage drop across the capacitor ($\frac{1}{C}Q$) is the same as the impressed voltage ($E(t)$) on the circuit. That is,

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

$$\Leftrightarrow \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E(t)}{R} \quad \text{(linear first-order DE)}$$

- NB. $I(t) = dQ(t)/dt$

  Example: 1) Suppose that in the LR-circuit of resistance is 12Ω, the inductance is $4H$ and a batter gives a constant voltage of $60v$. If the switch is closed, then find the current in the circuit at time t

- 2) A circuit loop contains a $5v$ batter, a 2Ω resistor and a $1F$ capacitor. Find the charge in circuit if $Q(0) = 0$. Find the current $I(t)$. (exercise)
  Solution: 1) Given $R = 12, L = 4, E = 60, I(0) = 0$
- Then $\frac{dI}{dt} + 4I = \frac{E(t)}{4}$ and we obtain $I(t) = 5 - 5e^{-3t}$

(D) Population Dynamics (Exponential Growth and Decay)

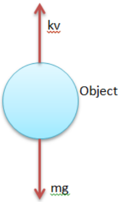
- The rate change of object y is proportional to the value of y. That is, $y' = ky$. (Reading homework)
  Example: Suppose the population count of colony of bacteria is 60 and after 2hrs 300. Find the total amount of the population at time t. (exercise)

(E) Falling Body

- Consider a falling object influenced by gravity and an air resistance proportional to the velocity of the object.

- Net force $F = mg + (-kv) = ma = mv'$

$$v' + \frac{k}{m}v = g \quad \text{(linear first-order DE)} \tag{13}$$



- NB. $a = v'$ and $v = s' \Leftrightarrow s = \int vdt$
  Example: A steel ball weighing 1 lb is dropped from 2500 ft with no velocity. As it fall air resistance is equal to $v/8$ in bounds. Find the limiting velocity and the time it takes for the ball to hit the ground.

Solution: Given $g = 32, kv = v/8, \omega = 1, v(0) = 0$

- Then we obtain $k = 1/8, mg = 1 \Rightarrow m = 1/32$
- Use formula eq. (13) we have $v' + 4v = 32 \Leftrightarrow v(t) = 8 - 8e^{-4t}$
- $s(t) = \int \left(8 - 8e^{-4t}\right) dt + c,\ \ s(0) = 0$
  $\Leftrightarrow 2500 = 8t + 2e^{-4t} - 2$
- Therefore, $\lim_{t\to\infty} v(t) = 8$ and $t \approx 312.5$sec

Summery of chapter one

- First-order DE $y' + p(x)y = q(x)$ can be
  -separable $(y' = g(x)h(y))$ or non-separable $(y' \neq g(x)h(y))$
  -homogeneous $(q(x) = 0)$ or non-homogeneous $(q(x) \neq 0)$
  -exact $(M_y = N_x)$ or non-exact $(M_y \neq N_x)$
  -linear or non-linear
- Non-separable DE $(y' = f(y/x)) \overset{\text{reduction}}{\rightarrow}$ separable DE, i.e., $y' = f(y/x) \overset{\text{reduction}}{\rightarrow} u' = (f(u) - u)/x$
- Integrating factor of first-order DE
  $F(x) = e^{\int ((M_y - N_x)N)dx}$ or $F(y) = e^{\int ((N_x - M_y)/M)dy}$
- Non-exact DE $\overset{\text{integrating factor}}{\rightarrow}$ Exact DE
- General solution of exact DE is:
  $f = \int Mdx + k(y) = c$ or $f = \int Ndy + l(x) = c$

...cont'

- General solution form of linear first-order DE $y' + p(x)y = q(x)$ is: $y(x) = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + c\right)$
- Non-linear (Bernoulli Eq.) $\overset{\text{transform}}{\rightarrow}$ linear DE, i.e., $y' + p(x)y = q(x)y^r \overset{\text{transform}}{\rightarrow} z' + (1-r)p(x)z = (1-r)q(x)$
- Application of first-order DE:
  -Mixing problem: $y' = \text{Input rate} - \text{Output rate}$
  -Newton's law of cooling: $T' = k(T - T_m)$
  -Electric series circuit:

  $$\text{LR: } LI' + RI = E \text{ and RC: } RQ' + 1/C\ Q = E$$

  -Exponential growth and decay: $y' = ky$
  -Falling body: $mg + (-kv) = mv' \Leftrightarrow v' + k/m\ v = g$

2.1 Homogeneous Linear Differential Equations of Second-Order

- A second-order ODE is linear if it can be written as:

$$y'' + p(x)y' + q(x)y = r(x) \tag{14}$$

  otherwise it is non-linear

- That is, eq. (14) is linear only if y and its derivatives have at most degree one as well as p, q and r are at most the function of x.

1. if $r(x) = 0$, then eq. (14) reduces to

$$y'' + p(x)y' + q(x)y = 0 \tag{15}$$

   which is called homogeneous second-order DE
2. if $r(x) \neq 0$, then eq. (14) is called non-homogeneous DE

...cont'

Example: Determine the following DE as linear/ non-linear, homogeneous/ non-homogeneous

1. $y'' + 25y = e^{-x}\cos x$. Ans. Linear non-homo. DE
2. $y'' + \frac{1}{x}y' + y = 0$. Ans. Linear homo. DE
3. $yy'' + (y')^2 = 0$. Ans. (Exercise)

Theorem (Principle of superposition). If y_1 and y_2 are solutions of eq. (15), then any linear combination of y_1 and y_2 (i.e., $y = c_1y_1 + c_2y_2$) is also a solution of eq. (15).

- Proof: Exercise
  Example: $y_1 = \cos x$ and $y_2 = \sin x$ are the solution of $y'' + y = 0$, then $y = c_1\cos x + c_2\sin x$ is also the solution. (verify)
- Ans. $y_1'' + y_1 \equiv 0$, $y_2'' + y_2 \equiv 0$
  We compute $y' = -c_1\sin x + c_2\cos x, y'' = -c_1\cos x - c_2\sin x$ and then by substitution we obtain $y'' + y \equiv 0$

...cont'

General and particular solution of DE

- A general solution of eq. (15) is $y = c_1 y_1 + c_2 y_2$ in which y_1 and y_2 are the solution of eq. (15) that are linearly independents.
- A particular solution of eq. (15) is obtained by assigning specific values to constants c_1 and c_2.

  Reduction of Order From Second into First:

  1. If second-order DE free of y or
  2. If second-order DE free of x

  Example: Solve the following DE

  1. $xy'' + 2y' = x^2 - 1$.
  2. $2yy'' - (y')^2 = 1$.
  3. $y'' + y' + y = 0$. (Exercise)

  Solution:

  1) $xy'' + 2y' = x^2 - 1$ is free of y. So by reduction of order:
- Let $y' = v \Leftrightarrow y'' = v'$. Then by substitution: we obtain $v' + \frac{2}{x}v = x - \frac{1}{x}$ which is first-order and linear

...cont'

- Thus, its general solution is given by
  $v = e^{-\int 2/xdx}\left(\int (x - \frac{1}{x})e^{\int 2/xdx}dx + c_1\right) = x^2/4 - 1/2 + c_1/x^2$
- But $y' = v = x^2/4 - 1/2 + c_1/x^2$
  $\Leftrightarrow y = \int (x^2/4 - 1/2 + c_1/x^2)dx + c_2$
- Therefore, $y = x^3/12 - x/2 - c_1/x + c_2$ is the general solution of the given DE
  2) $2yy'' - (y')^2 = 1$ is free of x. By reduction of order:
- Let $y' = v \Leftrightarrow y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}$, since $v \overset{dv/dy}{\rightarrow} y \overset{dy/dx}{\rightarrow} x$ (Chain rule)
- By substitution: $2yv'v - v^2 = 1$ which is first-order and non-linear, but it is separable DE
- So we have $\int \frac{2v}{v^2+1}dv = \int \frac{1}{y}dy + c_1 \Leftrightarrow v^2 + 1 = ye^{c_1}$
- But $y' = v = \pm\sqrt{ye^{c_1} - 1} \Leftrightarrow \int \frac{\pm 1}{\sqrt{ye^{c_1} - 1}}dy = \int dx + c_2$

- Hence, $y = e^{c_1}(x + c_2)^2 + \frac{1}{e^{c_1}}$ is the general solution

2.2 Homogeneous Linear DEs With Constant Coefficients

- Consider homogeneous linear DE with constant coefficients, say a, b, c:

$$ay'' + by' + cy = 0 \tag{16}$$

- Let $y = e^{\lambda x}$ be trial solution of eq. (16). Then $y' = \lambda e^{\lambda x},\ y'' = \lambda^2 e^{\lambda x}$
- By substituting y and its derivatives into eq. (16) we obtain $a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$. After simplification it becomes

$$a\lambda^2 + b\lambda + c = 0 \tag{17}$$

- Eq. (17) is called characteristic/ auxiliary equation of eq. (16).

- The roots of the characteristic eq. (16) are given by
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- Thus, $y = e^{\lambda_1 x}$ and $y = e^{\lambda_2 x}$ are the solutions of DE eq. (16)
- Eq. (17) may have three types of roots depending on the sign of the discriminant $b^2 - 4ac$ such that
- two distinct real roots if $b^2 - 4ac > 0$,
- double real roots if $b^2 - 4ac = 0$,
- complex conjugate roots if $b^2 - 4ac < 0$.
  Based on the roots, the general solution of DE eq. (16) may be three types:
  1. $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ when two distinct real roots
  2. $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$ when double real roots
  3. $y = e^{r}(c_1 \cos \omega x + c_2 \sin \omega x)$ when complex conjugate roots, where r is real part, ω is imaginary part

...cont'

- NB. To use characteristic equation method for solving the given DE we must check whether it is homogeneous linear with constant coefficients.

Example: Solve the following linear DE with constant coefficients

1. $2y'' - 8y = 0.$
2. $y'' + 6y' + 9y = 0.$
3. $y'' + 2y' + 5y = 0.$
4. $2y'' + 3y' = 0, y(0) = 1, y'(0) = 3.$ (exercise)

Solution:

- 1) $2y'' - 8y = 0$ is homo. linear with constant coefficients
- Characteristic equation is: $2\lambda^2 - 8 = 0$ since $a = 2, b = 0, c = 8$
- Its real roots are: $\lambda = \pm 2$ which are two distinct with $\lambda_1 = 2, \lambda_2 = -2$
- Hence, $y = c_1 e^{2x} + c_2 e^{-2x}$ is the general solution of the given DE

...cont'

- 2) $y'' + 6y' + 9y = 0$ is homo. linear with constant coefficients
- Characteristic equation is: $\lambda^2 + 6\lambda + 9 = 0$ since $a = 1, b = 6, c = 9$
- Its real roots are: $\lambda = -3$ which are double
- Hence, $y = c_1e^{-3x} + c_2xe^{-3x}$ is the general solution of the given DE
- 3) $y'' + 2y' + 5y = 0$ is homo. linear with constant coefficients
- Characteristic equation is: $\lambda^2 + 2\lambda + 5 = 0$ since $a = 1, b = 2, c = 5$
- Its roots are: $\lambda = -1 \pm 2i$ which are complex numbers with $r = -1, \omega = 2$
- Hence, $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$ is the general solution of the given DE

Euler-Cauchy Equation

- It is a homogeneous linear DE with variable coefficients of the form

$$Ax^2y'' + Bxy' + cy = 0 \quad (18)$$

- Let $y = x^m$ be a trial solution of Euler-Cauchy eq. (18). Then, $y' = mx^{m-1}, \ y'' = m(m-1)x^{-2}$.
- By substituting y and its derivatives into eq. (18) we get $Ax^2m(m-1)x^{-2} + Bxmx^{m-1} + cx^m = 0$
- This is simplified:

$$Am^2 + (B-A)m + c = 0 \quad (19)$$

- Eq. (19) is called characteristic equation of Euler-Cauchy eq. (18)
- The roots of eq. (19) are: $m = \dfrac{-(B-A) \pm \sqrt{(B-A)^2 - 4AC}}{2A}$

...cont'

- Thus, $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ are the solutions of eq. (18)
- Based on the types of roots of eq. (19) the general solution form of eq. (18) may have three types:

1. $y = c_1 x^{m_1} + c_2 x^{m_2}$
2. $y = c_1 x^m + c_2 (\ln x) x^m$
3. $y = x^r (c_1 \cos(k \ln x) + c_2 \sin(k \ln x))$,
   where r is real part, k is imaginary part

Example: Solve the following Euler-Cauchy equation

1. $x^2 y'' + 3/2 x y' - 1/2 y = 0.$
2. $x^2 y'' - 5 x y' + 9 y = 0.$
3. $x^2 y'' + x y' + 4 y = 0.$
4. $x^2 y'' - y = 0.$ (exercise)

...cont'

Solution:

- 1) $x^2y'' + 3/2xy' - 1/2y = 0$ is Euler-Cauchy equation
- By using formula from eq. (19) we obtain characteristic equation:
  $Am^2 + (B - A)m + C = 0, \;\; A = 1, B = 3/2, C = -1/2$
  $\Leftrightarrow m^2 + 1/2m - 1/2 = 0$
  $\Leftrightarrow m = 1/2$ or $m = -1$ which are two distinct real roots
- Hence, $y = c_1x^{1/2} + c_2x^{-1}$ is the general solution of the given DE
- 2) $x^2y'' - 5xy' + 9y = 0$ is Euler-Cauchy equation
- By using formula from eq. (19) we obtain characteristic equation:
  $Am^2 + (B - A)m + C = 0, \;\; A = 1, B = -5, C = 9$
  $\Leftrightarrow m^2 - 6m + 9 = 0$
  $\Leftrightarrow m = 3$ which is double root
- Hence, $y = c_1x^3 + c_2(\ln x)x^3$ is the general solution of the given DE

- 3) $x^2y'' + xy' + 4y = 0$ is Euler-Cauchy equation
- By using formula from eq. (19) we obtain characteristic equation: $Am^2 + (B - A)m + C = 0, \;\; A = 1, B = 1, C = 4$
  $\Leftrightarrow m^2 + 4 = 0$
  $\Leftrightarrow m = 0 \pm 2i$ which are conjugate complex roots
- Hence, $y = x^0(c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x))$ is the general solution of the given DE

Notice: If one solution y_1 of DE eq. (15) $(y'' + p(x)y' + q(x)y = 0)$ is given, then the second solution y_2 is given as:

$$y_2 = y_1 \int \left(\frac{1}{y_1^2} e^{-\int p(x)dx} \right) dx \tag{20}$$

...cont'

Example: Find y_2 and give the general solution of the following DE

1. $(x^2 - x)y'' - xy' + y = 0$; $y_1 = x$.
2. $x^2y'' - 2xy' + 2y = 0$; $y_1 = x^2$. (exercise)

Solution:

- 1) First rewrite the given DE as eq. (15):
$$y'' - \frac{1}{x-1}y' + \frac{1}{x^2 - x}y = 0 \Rightarrow p(x) = -\frac{1}{x-1}$$
- From eq. (20) y_2 is given as:
$$y_2 = y_1 \int \left(\frac{1}{y_1^2} e^{-\int p(x)dx} \right) dx = x \int \left(\frac{1}{x^2} e^{\int \frac{1}{x-1} dx} \right) dx = x \ln x + 1$$
- Therefore, $y_2 = x \ln x + 1$ and the general solution is
$$y = c_1 x + c_2(x \ln x + 1)$$
- 2) Ans. $y_2 = -x$ and $y = c_1 x^2 - c_2 x$

2.3 Method For Solving Non-Homogeneous Linear DEs

- Consider the non-homogeneous linear DE

$$y'' + p(x)y' + q(x)y = r(x) \tag{21}$$

- The corresponding homogeneous DE of eq. (21) is

$$y'' + p(x)y' + q(x)y = 0 \tag{22}$$

- A general solution of eq. (21) is the solution of the form

$$y = y_h + y_p, \tag{23}$$

where y_h is homogeneous solution of eq. (22),
y_p is the particular solution of eq. (21).

- To solve the non-homogeneous eq. (21), we have to solve the homogeneous eq. (22) to get y_h first, and then find the particular solution y_p so that we obtain the general solution eq. (23) of eq. (21).

We have two methods to solve eq. (1):

1. Method of undetermined coefficient
2. Method of variation of parameter

1) Method of undetermined coefficient

- This method is suitable for linear DEs with constant coefficients a and b:

$$y'' + ay' + by = r(x), \tag{24}$$

where $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions.

Table: Term in $r(x)$ and choose for y_p

Term in $r(x)$	**Choose for** y_p
$ke^{\lambda x}$	$ce^{\lambda x}$
kx^n	$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$
$k\cos\omega x$	$M\cos\omega x + N\sin\omega x$
$k\sin\omega x$	$M\cos\omega x + N\sin\omega x$
$ke^{\lambda x}\cos\omega x$	$e^{\lambda x}(M\cos\omega x + N\sin\omega x)$
$ke^{\lambda x}\sin\omega x$	$e^{\lambda x}(M\cos\omega x + N\sin\omega x)$
$kx^n e^{\lambda x}$	$(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)e^{\lambda x}$

- There are three choice rule for method of undetermined coefficient:
  (a) Basic rule
  (b) Modification rule
  (c) Sum rule

- a) Basic Rule: If the right function $r(x)$ in eq. (24) is one of the functions in the first column in the above table, choose y_p in the same line and determine coefficients by substituting y_p and its derivatives into eq. (24).
- b) Mollification Rule: If a term in your choice for y_p happens to be a solution of homogeneous DE corresponding to eq. (24) multiply this term by x (or if the characteristic equation of homogeneous DE has double root, multiply the term by x^2). Then determine the coefficients by substituting y_p and its derivatives into eq. (24).
- c) Sum Rule: If $r(x)$ in eq. (24) is a sum of functions in the first column of the table, choose y_p in the corresponding lines of the second column.

Example: Solve the following non-homogeneous DE

1. $y'' + y = x^2$.
2. $y'' - 4y = e^{2x}$.
3. $y'' + 2y' + y = 10e^{-x}$.
4. $y'' + 2y' + 5y = e^x + \cos 2x - \sin 2x$.

Solution:

- 1) $y'' + y = x^2$ is non-homo. DE with constant coefficients $\Rightarrow$To solve this DE method of undetermined coefficients is applied
- Homo. DE $y'' + y = 0$ and its chara. eq. is $\lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$ $\Rightarrow y_h = c_1 \cos x + c_2 \sin x$
- Right function $r(x) = x^2$. The second line in the table suggests that the choice of $y_p = k_2x^2 + k_1x + k_0$ (basic rule).
- Then $y_p' = 2k_2x + k_1, y_p'' = 2k_2$ and by substitution: $y_p'' + y_p = 2k_2 + k_2x^2 + k_1x + k_0 = x^2 \Leftrightarrow k_2 = 1, k_1 = 0, k_0 = -2$

...cont'

- Thus $y_p = x^2 - 2$
- Therefore, $y = y_h + y_p = c_1 \cos x + c_2 \sin x + x^2 - 2$ is the general solution of the given DE
- 2) Corresponding hom. DE is $y'' - 4y = 0$, chara. eqn. is $\lambda^2 - 4 = 0 \Leftrightarrow \lambda = \pm 2$
- Hom. sol. is $y_h = c_1 e^{2x} + c_2 e^{-2x} \Rightarrow y_{h_1} = e^{2x},\ y_{h_2} = e^{-2x}$
- The right function is $r(x) = e^{2x}$ which is the solution of homo. DE.
- Thus, we choose $y_p = cxe^{2x}$ (mollification rule)
- By substituting y_p and its derivatives into the given DE we get $c = 1/4$.
- Thus, $y_p = 1/4\ xe^{2x}$
- Hence $y = c_1 e^{2x} + c_2 e^{-2x} + 1/4\ xe^{2x}$ is the general solution

- 3) Corresponding hom. DE is $y'' + 2y' + y = 0$, chara. eqn. is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow \lambda = -1$ which is double root
- Hom. sol. is $y_h = c_1 e^{-x} + c_2 x e^{-x}$
- Right function is $r(x) = 10e^{-x}$ and chara. eqn. has double root.
- Thus, we choose $y_p = cx^2 e^{-x}$ (mollification rule)
- By substituting y_p and its derivatives into the given DE we get $c = 5$.
- Thus, $y_p = 5x^2 e^{2x}$
- Hence, $y = c_1 e^{-x} + c_2 x e^{-x} + 5x^2 e^{2x}$ is the general solution

...cont'

- 4) Corresponding hom. DE is $y'' + 2y' + 5y = 0$, chara. eqn. is $\lambda^2 + 2\lambda + 5 = 0 \Leftrightarrow \lambda = -1 \pm 2i$
- Hom. sol. is $y_h = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$
- Right function is $r(x) = e^x + \cos 2x - \sin 2x$ and chara. eqn. has double root.
- Thus, we choose $y_p = ce^x + M \cos 2x + N \sin 2x$ (sum rule)
- By substituting y_p and its derivatives into the given DE we get $c = 1/8, M = 5/17, N = 3/17.$
- Thus, $y_p = 1/8\ e^{-x} + 5/17\ \cos 2x + 3/17\ \sin 2x$
- Hence, $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + 1/8\ e^x + 5/17\ \cos 2x + 3/17\ \sin 2x$ is the general solution

2) Method of Variation of Parameter (More general method than UC)

- Consider non-homogeneous differential eq. (21). The particular solution y_p of eq. (21) is given by

$$y_p = -y_1 \int \left(\frac{y_2 r(x)}{W(y_1, y_2)} \right) dx + y_2 \int \left(\frac{y_1 r(x)}{W(y_1, y_2)} \right) dx, \tag{25}$$

  where y_1 and y_2 are the solutions of the corresponding homo. differential eq. (22) and the Wronskian of y_1 and y_2 is given as:

$$W(y_1, y_2) = \det\{(y_1, y_2), (y_1', y_2')\} = y_1 y_2' - y_1' y_2$$

- Hence, the general solution of eq. (21) is $y = c_1 y_1 + c_2 y_2 + y_p$.
- Notice: $W(y_1, y_2) = 0 \Rightarrow y_1$ and y_2 are linear dependent.
  $W(y_1, y_2) \neq 0 \Rightarrow y_1$ and y_2 are linear independent.

Example: Solve the following DE

1. $x^2y'' - 2y = 1/x$.
2. $y'' + y = \sec x$.
3. $y'' - 9y = -x$. (exercise)

Solution:

- 1) Corresponding homo. DE: $x^2y'' - 2y = 0$ -Euler Cauchy eqn.
- Chara. eqn. : $m^2 - m - 2 = 0 \Leftrightarrow m = 2$ or $m = -1$
  $\Rightarrow y_1 = x^2, y_2 = x^{-1}$ and $y_h = c_1x^2 + c_2x^{-1}$,
  $W(y_1, y_2) = \det\left\{(x^2, x^{-1}), (2x, -1/x^2)\right\} = -3$
- Right function is: $r(x) = 1/x^3$ since the given DE $x^2y'' - 2y = 1/x$ can be written as eq. (21) form: $y'' - 2/x^2\ y = 1/x^3$
- By applying formula eq. (25) we obtain $y_p = -1/9x - \ln x/3x$
- Therefore, $y = c_1x^2 + c_2x^{-1} - 1/9x - \ln x/3x$ is the general solution

- 2) Corresponding homo. DE: $y'' + y = 0$
- Chara. eqn. : $\lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$
  $\Rightarrow y_1 = \cos x, y_2 = \sin x$ and $y_h = c_1 \cos x + c_2 \sin x$,
  $W(y_1, y_2) = \det\{(\cos x, \sin x), (-\sin x, \cos x)\} = 1$
- Right function is: $r(x) = \sec x$
- By applying formula from eq. (25) we obtain
  $y_p = \cos x \ln|\cos x| + x \sin x$
- Therefore, $y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$ is the general solution

2.3 System of Differential Equations

- It involves two or more equations that contain derivatives of two or more dependent variables wrt a single independent variable.
- System of DEs can be solved by i) substitution method, ii) elimination method or iii) eigen value-eigen vector method

Example: Solve system of DE

1. $$\begin{cases} \dfrac{dx}{dt} = 3y, \\ \dfrac{dy}{dt} = 2x. \end{cases}$$

2. $$\begin{cases} \dfrac{dx}{dt} - 3y + 16\cos t = 0, \\ x + 3\dfrac{dy}{dt} = 0. \end{cases}$$

...cont'

Solution:

- 1)

$$\begin{cases} \dfrac{dx}{dt} = 3y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \\ \dfrac{dy}{dt} = 2x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \end{cases}$$

- (1) $\Leftrightarrow x'' = 3y'$ but $y' = 2x$ from (2)
  Then, we obtain $x'' - 6x = 0$ (substitution method)
  $\Leftrightarrow x(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$
- From (1) we have $y = x'/3$
  $\Leftrightarrow y(t) = \sqrt{6}/3\ c_1 e^{\sqrt{6}t} - \sqrt{6}/3\ c_2 e^{-\sqrt{6}t}$
- Hence

$$\begin{cases} x(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} \\ y(t) = \sqrt{6}/3\ c_1 e^{\sqrt{6}t} - \sqrt{6}/3\ c_2 e^{-\sqrt{6}t} \end{cases}$$

  is the general solution of the system of DE

- 2)

$$\begin{cases} \dfrac{dx}{dt} - 3y + 16\cos t = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \\ x + 3\dfrac{dy}{dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \end{cases}$$

- (1) $\Leftrightarrow x'' - 3y' - 16\sin x = 0$ but $y' = -x/3$ from (2)
  Then, we obtain $x'' + x - 16\sin x = 0$ (substitution method)
  $\Leftrightarrow x(t) = c_1 \cos t + c_2 \sin t - 8t\cos t$
- From (1) we have $y = (x' + 16\cos t)/3$
  $\Leftrightarrow y(t) = (-c_1 \sin t + c_2 \cos t + 8\cos t + 8t\sin t)/3$
- Hence

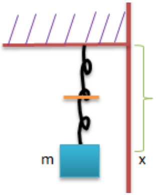
$$\begin{cases} x(t) = c_1 \cos t + c_2 \sin t - 8t\cos t \\ y(t) = (-c_1 \sin t + c_2 \cos t + 8\cos t + 8t\sin t)/3 \end{cases}$$

  is the general solution

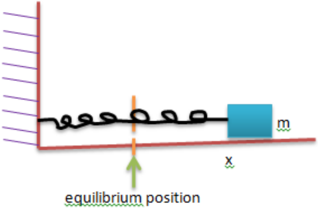
2.5 Applications of Second-Order Differential Equations

A) Vibrating Spring

- We consider the motion of an object with mass m at the end of a spring that is ether vertical or horizontal on a lever surface.



(a)



(b)

- Hooke's law: If the spring is stretched or compressed x units from its natural length, then it exerts a force that is proportional to x, i.e., restoring force $F = -kx$, where k is spring constant.

...cont'

Vibrating spring take place at frictional force or no frictional force:

- (i) Assuming that there is no frictional force, the Newton's second-law implies $mx'' + kx = 0$. (justify!) $F = ma = -kx$, but $a = x''$
- The position of the mass m is $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ or $x(t) = A \cos(\omega t + \delta)$,
  where $\omega = \sqrt{k/m}$ is natural frequency of the vibration,
  $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude of max displacement of mass from equilibrium,
  $\delta = \tan^{-1}(c_2/c_1)$ is phase angle,
  $T = 2\pi/\omega = 2\pi\sqrt{m/k}$ is called the period.
- (ii) Assuming that there is frictional force, the Newton's second-law gives $mx'' + cx' + kx = 0$. (justify!)
  $F =$ restoring force $+$ damping force
  $\Leftrightarrow mx'' + cx' + kx = 0$, where c is damping constant
- Chara. eqn. $m\lambda^2 + c\lambda + k = 0 \Leftrightarrow \lambda = (-c \pm \sqrt{c^2 - 4mk})/2m$

- The position of the mass m can be three types depending on the sign of discriminant: $c^2 - 4mk > 0$ (over damping), $c^2 - 4mk = 0$ (critical damping), $c^2 - 4mk < 0$ (under damping)

Example:

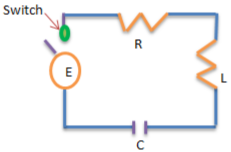
1. A spring with a mass of 2kg has natural length 0.5m. A force of 25.6N is required to stretched to a length of o.7m and then released with initial velocity om/s, find the position of the mass at any time t. Find also amplitude, frequency, phase angle and period of vibration.
2. Suppose that the spring given on example 1 is immersed in a fluid with damping constant 40. Find the position of the mass at t if it starts from equilibrium position and is given a push to start it with an initial velocity is 0.6m/s

Example:

- 1) Given $F = 25.6, m = 2, x_e = 0.5, x_f = 0.7, v(0) = 0 = x'(0)$
  $25.6 = F = kx = k(0.7 - 0.5) = 0.2k \Rightarrow k = 128$
- Thus $mx'' + kx = 0$ (frictional force is neglected)
  $\Leftrightarrow 2x'' + 128x = 0 \Leftrightarrow x(t) = c_1 \cos 8t + c_2 \sin 8t$
  $x(0) = 0.2 \Rightarrow c_1 = 0.2$ and $x'(0) = 0 \Rightarrow c_2 = 0$
- Hence, the position of the mass at any time t is $x(t) = 0.2 \cos 8t$,
  $A = 0.2$ is amplitude, $\omega = 8$ is frequency, $\delta = 0$ is phase angle,
  $T = \pi/4$ is period
- 2) Given $c = 40, m = 2, k = 128, x(0) = 0, x'(0) = 0.6 = v(0)$
- Thus $2x'' + 40x' + 128x = 0$ (there is frictional force)
  Chara. eqn. $\lambda^2 + 20\lambda + 64 = 0$
  $\Rightarrow \lambda = -4$ or $\lambda = -16$ (over damping) and $x(t) = c_1 e^{-4t} + c_2 e^{-16t}$
  $x(0) = 0, x'(0) = 0.6 \Rightarrow c_1 = 0.05, c_2 = -0.5$
- Hence, the position of the mass at any time t is
  $x(t) = 0.05e^{-4t} - 0.5e^{-16t}$

B) Electric RLC-Circuits

- Kirchhoff's voltage law says that the sum of voltage drops across inductor (LI'), resistor (RQ') and capacitor (Q/C) is equal to the supplied voltage, i.e., $LQ'' + RQ' + Q/C = E(t)$.



Example: Find the charge and current at time t in RLC-circuit if $R = 40\Omega, L = 1H, C = 16 \times 10^{-4}F, E(t) = 100\cos 10t$ and the initial charge and current are both zero.

- Solution: Given $R = 40\Omega, L = 1H, C = 16 \times 10^{-4}F, E(t) = 100\cos 10t, Q(0) = 0, I(0) = 0 = Q'(0)$
- Thus, we get DE: $Q'' + 40Q' + 625Q = 100\cos 10t$

...cont'

- Homo. DE: $Q'' + 40Q' + 625Q = 0$
- Chara. eqn. : $\lambda^2 + 40\lambda + 625 = 0 \Leftrightarrow \lambda = -20 \pm 15i$
  $\Leftrightarrow Q_h(t) = e^{-20t}(c_1 \cos 15t + c_2 \sin 15t)$
- Since $r(t) = 100 \cos 10t$, we choose $Q_p(t) = M \cos 10t + N \sin 10t$ (by basic rule)
- By substituting Q_p and its derivatives into the obtained DE we get $M = 84/697, N = 64/697$
- Then,
  $Q(t) = e^{-20t}(c_1 \cos 15t + c_2 \sin 15t) + 84/697\ \cos 10t + 64/697\ \sin 10t$
- But $Q(0) = 0, Q'(0) = 0 \Rightarrow c_1 = -84/697, c_2 = -464/2091$
- Hence, $Q(t) = Q_h(t) + Q_p(t)$ and $I(t) = Q'(t)$

...cont'

Summary of chapter two

- Principle of superposition: If y_1 and y_2 are solutions of homo. second-order DE $\Rightarrow y(t) = c_1y_1 + c_2y_2$ is also the solution.
- Reduction of order is applied in solving DE when second-order DE free of y: (setting $y' = v, y'' = v'$) or
  when second-order DE free of x: (setting $y' = v, y'' = v'v$)
- Homo. linear second-order DE with constant coefficients: $ay'' + by' + cy = 0$ can be solved by chara. eqn. : $a\lambda^2 + b\lambda + c = 0$.
- Euler-Cauchy homo. linear DE with variable coefficients: $ay'' + by' + cy = 0$ can be solved by chara. eqn. :
  $$Am^2 + (B - A)m + C = 0.$$
- If the solution y_1 of homo. linear DE: $y'' + p(x)y' + q(x)y = 0$ is given, then solution: $y_2 = y_1 \int \left(\frac{1}{y_1} e^{-\int p(x)dx}\right) dx$

- Non-homo. second-order DE can be solved:
  (i) undetermined coefficient method: Particular solution y_p be chosen by basic, mollification and sum rule
  (ii) variation of parameter method: Particular solution y_p is:
  $$y_p = -y_1 \int \left(\frac{y_2 r(x)}{W(y_1, y_2)} \right) dx + y_2 \int \left(\frac{y_1 r(x)}{W(y_1, y_2)} \right) dx, \tag{26}$$
  Wronskian of y_1 and y_2 is given as:
  $$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$
- Applications of second-order DEs:
  - Vibrating spring
  Hooke's law: $F = -kx$
  Newton's second-law: $mx'' + kx = 0$ (no frictional force)
  $mx'' + cx' + kx = 0$ (there is frictional force)
  -Electric RLC-circuit: Kirchhoff's voltage law:
  $$LQ'' + RQ' + Q/C = E(t)$$

Chapter Three: Laplace Transform

- It is a powerful method for solving linear ODEs and corresponding IVPs as well as systems of DEs
- Besides being a different and efficient alternative to undetermined coefficients and variation of parameters, the laplace method is advanageous for input terms that are piecewise defined, periodic or impulsive

  3.1 Laplace Transformations
- The laplace transform of a function $f(t)$ defined for $0 \leq t < \infty$ is denoted by $F(s)$ and given by
  $F(s) = l(f(t)) = \int_0^\infty e^{-st} f(t) dt$
- The function $f(t)$ is the inverse transform of $F(s)$, i.e.,
  $f(t) = l^{-1}(F(s)) = l^{-1}(\int_0^\infty e^{-st} f(t) dt)$
- The laplace transform is a linear operator, i.e.,
  $l(af(t) + bg(t)) = al(f(t)) + bl(g(t))$

...cont'

Example: Find the laplace transform of

1. $f(t) = k$.
2. $f(t) = e^{at}$.
3. $f(t) = \cosh at$.
4. $f(t) = \sinh at$. (exercise)

Solution:

- 1) $f(t) = k$ is defined on $[0, \infty)$.
  $F(s) = \int_0^\infty e^{-st} k dt = \lim_{b \to \infty} \int_0^b e^{-st} k dt = -\dfrac{ke^{-st}}{s}|_0^\infty = \dfrac{k}{s}$
- 2) $F(s) = \int_0^\infty e^{-st} e^{at} dt = -\dfrac{e^{-(s-a)t}}{s-a}|_0^\infty = \dfrac{1}{s-a}$
- 3) $F(s) = \int_0^\infty e^{-st} \cosh at dt = \int_0^\infty e^{-st} \dfrac{e^{at} + e^{-at}}{2} dt =$
  $\dfrac{1}{2(s-a)} + \dfrac{1}{2(s+a)} = \dfrac{s}{s^2 - a^2}$

...cont'

Table: Laplace transform of some functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
k	$\frac{k}{s}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
e^{at}	$\frac{1}{s-a}$	e^{-at}	$\frac{1}{s+a}$

Laplace transform techniques:

- S-shifting and T-shifting

- S-shifting: If $f(t)$ has transform $F(s)$, then $e^{at}f(t)$ has the transform $F(s-a)$, i.e., $l(f(t)) = F(s) \Rightarrow l(e^{at}f(t)) = F(s-a)$ or $e^{at}f(t) = l^{-1}(F(s-a))$.

Example 1: Find the laplace transform of

1. $f(t) = te^{2t} + e^{-t}\sin 6t$
2. $f(t) = 4e^{-3t} + e^{t}\cosh 2t$

Solution: (Apply S-shifting method)

- 1) Since $l(t) = \dfrac{1}{s^2}$ and $l(\sin 6t) = \dfrac{6}{s^2+36}$, then
$$l(f(t)) = \frac{1}{(s-2)^2} + \frac{6}{(s+1)^2+36}$$
- 2) Since $l(4) = \dfrac{1}{s}$ and $l(\cosh 2t) = \dfrac{s}{s^2-4}$, then
$$l(f(t)) = \frac{1}{(s+3)} + \frac{s}{(s-1)^2-4}$$

Example 2: Find the inverse laplace transform of

1. $F(s) = \dfrac{s+16}{s^2-16}$.
2. $F(s) = \dfrac{1}{s^2-4s+4}$.
3. $F(s) = \dfrac{1}{s^2+3} - \dfrac{1}{s+3}$. (Exercise)

Solution: (Apply S-shifting method)

- 1) $F(s) = \dfrac{s}{s^2-4^2} + 4\dfrac{4}{s^2-4^2} \Leftrightarrow f(t) = \cosh 4t + 4\sinh 4t$
- 2) $F(s) = \dfrac{1}{(s-2)^2} \Leftrightarrow f(t) = te^{2t}$

3.2. Differential and integration of laplace of transform

- Differential of laplace of transform: If $F(s) = \int_0^\infty e^{-st} f(t)dt$, then $F'(s) = -\int_0^\infty e^{-st} tf(t)dt = -l(tf(t))$

...cont'

- Thus, $-tf(t) = l^{-1}(F'(s))$

Example: Find the inverse transform of $F(s) = \ln(1 + \frac{\omega^2}{s^2})$

Solution:

- $F'(s) = (\ln(s^2 + \omega^2) - \ln s^2)' = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}$

  $\Leftrightarrow -tf(t) = l^{-1}(\frac{2s}{s^2 + \omega^2}) - l^{-1}(\frac{2}{s}) = 2\cos\omega t - 2$
- Therefore, $f(t) = \frac{2\cos\omega t - 2}{-t}$
- Integration of laplace transform: $\int_s^\infty F(\bar{s})d\bar{s} = l(\frac{f(t)}{t})$ (justify)
- Justification: $\int_s^\infty F(\bar{s})d\bar{s} = \int_s^\infty (\int_0^\infty e^{-st} f(t)dt)d\bar{s} =$

  $\int_0^\infty f(t)(\int_s^\infty e^{-st} d\bar{s})dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = l(\frac{f(t)}{t})$

- Thus, $\dfrac{f(t)}{t} = l^{-1}(\int_s^{\infty} F(\bar{s})d\bar{s})$
- Laplace transform of integral: $l(\int_0^t f(r)dr) = \dfrac{F(s)}{s}$, where $f(t)$ is piecewise continuous function for $t \geq 0$
- Thus, $\int_0^t f(r)dr = l^{-1}(\dfrac{F(s)}{s})$

Example: Find the inverse transform of $F^*(s) = \dfrac{1}{s(s^2+\omega^2)}$

Solution:

- Let $F(s) = \dfrac{1}{s^2+\omega^2} = \dfrac{1}{\omega}\dfrac{\omega}{s^2+\omega^2} \Rightarrow f(t) = l^{-1}(F(s)) = \dfrac{\sin \omega t}{\omega}$
- $l^{-1}(F^*(s)) = l^{-1}(\dfrac{F(s)}{s}) = \int_0^t f(r)dr = \int_0^t \dfrac{\sin \omega r}{\omega}dr = \dfrac{1-\cos \omega t}{\omega^2}$

...cont'

- Laplace transform of derivatives: The transform of first and second derivatives of $f(t)$ are given by
  $l(f'(t)) = sl(f(t)) - f(0)$ (justify)
  $l(f''(t)) = s^2 l(f(t)) - sf(0) - f'(0)$ (justify)
- Justification:
  $$l(f'(t)) = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t) + s\int_0^\infty e^{-st} f(t)dt \quad \text{(by part)}$$
  $$= -f(0) + sl(f(t))$$
  Examlpe: Let $f(t) = e^{2t}$, then $l(f'(t)) = sl(f(t)) - f(0) = \dfrac{2}{s-2}$

Application of laplace transform: To solve ODE we must follow three steps:

i) ODE transform $\Rightarrow$ algebraic equation
ii) Algebraic equation $\Rightarrow$ solve
iii) Inversing the laplace transform resulting the solution

Example: Solve

1. IVP: $y'' - y = t,\ y(0) = 1, y'(0) = 1$
2. system of DE:

$$\begin{cases} x' = y, \\ y' = x,\ x(0) = 1, y(0) = 0 \end{cases}$$

Solution:

- 1) $l(y'') - l(y) = l(t)$
  $\Leftrightarrow s^2 l(y) - sy(0) - y'(0) - l(y) = \dfrac{1}{s^2}$
  $\Leftrightarrow (s^2 - 1)l(y) - s - 1 = \dfrac{1}{s^2}$
  $\Leftrightarrow l(y) = \dfrac{1}{s+1} + \dfrac{1}{s^2 - 1} - \dfrac{1}{s^2}$
  Therefore, $y(t) = e^{-t} + \sinh t - t$ is the solution of the given IVP.

- 2)

$$\begin{cases} l(x') = l(y), \\ l(y') = l(x) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} sl(x) - x(0) = l(y), \\ sl(y) - y(0) = l(x) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} sl(x) - 1 = l(y), \\ sl(y) - 0 = l(x) \end{cases}$$

$\Rightarrow s^2 l(y) - 1 = l(y)$

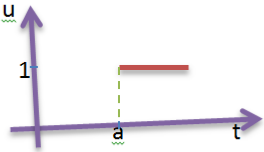
$\Leftrightarrow l(y) = \dfrac{1}{s^2 - 1}$

- Hence, $x(t) = \cosh t$ and $y(t) = \sinh t$

...cont'

- Unit step function: It is defined to be

$$u(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$



- Unit step function is used to write piecewise defined function in compact form

  Example:

$$1)\ f(t)=\begin{cases}g(t), & 0\le t<a\\ h(t), & t\ge a\end{cases} \quad = g(t)-g(t)u(t-a)+h(t)u(t-a)$$

$$2)\ f(t)=\begin{cases}0, & 0\le t<a\\ g(t), & a\le t<b\\ 0, & t\ge b\end{cases}\quad = g(t)(u(t-a)-u(t-b))$$

- Laplace transform of unit step function:
  $l(u(t))=\dfrac{1}{s}$ and $l(u(t-a))=\dfrac{e^{-as}}{s}$
- T-shifting method: If $l(f(t))=F(s)$, then for $a>0$,
  $l(f(t-a)u(t-a))=e^{-as}F(s)$ or $l(f(t)u(t-a))=e^{-as}l(f(t+a))$
- Its inverse $f(t-a)u(t-a)=l^{-1}(e^{-as}F(s))$ or
  $f(t)u(t-a)=l^{-1}(e^{-as}l(f(t+a)))$, respectively

Example 1) Find the laplace transform of

$$a)\ f(t)=\begin{cases} t,\ 0\leq t<1 \\ 0,\ t\geq 1 \end{cases}$$

$$b)\ f(t)=\begin{cases} 0,\ 0\leq t<2 \\ t^2,\ t\geq 2 \end{cases}$$

Solution:

- a) First rewrite f in compact form interms of unit step function:

$$f(t)=\begin{cases} t,\ 0\leq t<1 \\ 0,\ t\geq 1 \end{cases} = t-tu(t-1)$$

- $l(f(t))=l(t-tu(t-1))=\frac{1}{s^2}-e^{-s}l(t+1)=\frac{1}{s^2}-e^{-s}(\frac{1}{s^2}+\frac{1}{s})$

- b) Rewrite f in compact form:

$$f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ t^2, & t \geq 2 \end{cases} = t^2 u(t-2)$$

- $l(f(t)) = l(t^2 u(t-2)) = e^{-2s} l(t+2)^2 = e^{-2s} l(t^2 + 4t + 4) = e^{-2s}(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s})$

  Examlpe 2: Find inverse transform of $\frac{e^{-2s}}{s-4}$

  Solution:

- Let $F(s) = \frac{1}{s-4}$, then $l^{-1}(F(s)) = e^{4t}$. From laplace transform e^{-2s} we have unit step function u centered at $a = 2$
- Then $l^{-1}(\frac{e^{-2s}}{s-4}) = e^{4(t-2)} u(t-2)$

...cont'

Example 3: Solve $y' + y = f(t),\ y(0) = 5$ where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

Solution:

- $sl(y) - y(0) + l(y) = l(3\cos t u(t-\pi))$
  $\Leftrightarrow (s+1)l(y) - 5 = 3e^{-\pi s} l(\cos(t+\pi))$
  $\Leftrightarrow (s+1)l(y) = 5 + 3e^{-\pi s}(l(\cos(t)\cos\pi) - l(\sin(t)\sin\pi))$
  $\Leftrightarrow (s+1)l(y) = 5 - \dfrac{3s}{s^2+1}e^{-\pi s}$
  $\Leftrightarrow l(y) = \dfrac{5}{s+1} - \dfrac{3s}{(s+1)(s^2+1)}e^{-\pi s}$
  $= \dfrac{5}{s+1} - \dfrac{3e^{-\pi s}}{2}\left(\dfrac{-1}{s+1} + \dfrac{1}{s^2+1} + \dfrac{s}{s^2+1}\right)$
- Hence, $y(t) = 5e^{-t} - \dfrac{3u(t-\pi)}{2}(e^{-(t-\pi)} + \sin(t-\pi) + \cos(t-\pi))$

- Laplace transform of special linear ODEs with variable coefficients:
- $l(ty') = -\frac{d}{ds}(sl(y) - y(0)) = -l(y) - s\frac{dl(y)}{ds}$
- $l(ty'') = -\frac{d}{ds}(s^2 l(y) - sy(0) - y'(0)) = -2sl(y) - s^2\frac{dl(y)}{ds} + y(0)$
  Example: solve $ty'' + (1-t)y' + y = 0$
- Solution: $l(ty'') + l(y') - l(ty') + l(y) = 0$
  $\Leftrightarrow -2sl(y) - s^2\frac{dl(y)}{ds} + y(0) + sl(y) - y(0) + l(y) + s\frac{dl(y)}{ds} + l(y) = 0$
  $\Leftrightarrow (-s^2 + s)\frac{dl(y)}{ds} + (-s+2)l(y) = 0 \Leftrightarrow \frac{dl(y)}{l(y)} = \frac{s-2}{-s^2+s}ds$
  $\Leftrightarrow \ln l(y) = \ln(s-1) - 2\ln s \Leftrightarrow y(t) = 1 - t$
- Shifting data problem, after the problem is shifted to zero then solve it, example: $y'' + y = 2t,\ y(\pi/4) = \pi/2,\ y'(\pi/4) = 2 - \sqrt{2}$
- Let $t = \bar{t} + \pi/4$, then
  $\bar{y}'' + \bar{y} = 2(\bar{t} + \pi/4),\ \bar{y}(0) = \pi/2,\ \bar{y}'(0) = 2 - \sqrt{2},\ \bar{t} = t - \pi/4$

3.3 Convolution and integral equations

- The convolution of the functions f and g is defined by integral: $(f*g)(t) = \int_0^t f(r)g(t-r)dr$
  Property:
- $l(f*g) = l(f)l(g)$
- $f*g = l^{-1}(l(f)) * l^{-1}(l(g))$ is inverse transform
- $f*g = g*f \;\; \cdots$ commutaive
  Example 1: Find the convolution of f and g if
  a) $f = e^t, g = 1$ b) $f = t, g = e^{2t}$
  Ans. a) $f*g = e^t - 1$ b) $f*g = -t/2 - 1/4 + e^{2t}/4$
  Example 2: Find function f if $F(s) = \dfrac{1}{(s-3)(s+4)}$
  Ans. $f(t) = l^{-1}\left(\dfrac{1}{s-3}\right) * l^{-1}\left(\dfrac{1}{s+4}\right) = e^{3t} * e^{-4t} = \dfrac{1}{7}(e^{3t} - e^{-4t})$

Integral equations

- Convolution helps in solving certain integral equations, that is, equations in which the unknown function $y(t)$ appears in an integral of the form of convolution

  Example: Solve the following integral equations

  1. $y(t) - \int_0^t y(r)\sin(t-r)dr = t$
  2. $y(t) - \int_0^t (1+r)y(t-r)dr = 1 - \sinh t$

  Solution:

- 1) $l(y) - l(y * \sin t) = l(t)$

  $\Leftrightarrow l(y) - \dfrac{1}{s^2+1} l(y) = \dfrac{1}{s^2}$

  $\Leftrightarrow y = l^{-1}(\dfrac{1}{s^2}) + l^{-1}(\dfrac{3!}{3! s^{3+1}})$

  $\Leftrightarrow y(t) = t + \dfrac{1}{6}t^3$

- 2) $l(y) - l((1+t)*y(t)) = l(1-\sinh t)$
  $\Leftrightarrow l(y) - (\frac{1}{s} + \frac{1}{s^2})l(y) = \frac{1}{s} - \frac{1}{s^2-1}$
  $\Leftrightarrow y = l^{-1}(\frac{s}{s^2-1})$
  $\Leftrightarrow y(t) = \cosh t$

Summary of Chapter Three

- Laplace transform: $F(s) = \int_0^\infty e^{-st} f(t)dt$
- Inverse laplace transform: $f(t) = l^{-1}(F(s))$
- Technique of laplace transform:
  1. s-shifting: $l(f(t)) = F(s) \Leftrightarrow l(e^{-as}f(t)) = F(s-a)$
  2. t-shifting: $l(f(t)) = F(s) \Leftrightarrow l(f(t-a)u(t-a)) = e^{-as}F(s)$ or $l(f(t)u(t-a)) = e^{-as}l(f(t+a))$
- Piecewise defined function in compact form:

$$f(t) = \begin{cases} g(t), & a \le t < b \\ h(t), & t \ge b \end{cases} = g(t)(u(t-a) - u(t-b)) + h(t)u(t-b)$$

- Derivative of laplace transform: $F'(s) = -l(tf(t))$
- Integration of laplace transform: $\int_s^\infty F(\bar{s})d\bar{s} = l\left(\frac{f(t)}{t}\right)$
- Laplace transform of 1$^{\text{st}}$ derivative:
  $l(y'(t)) = sl(y(t)) - y(0)$ or $l(ty'(t)) = -l(y(t)) - s\frac{d}{ds}l(y(t))$
- Laplace transform of 2$^{\text{nd}}$ derivative:
  $l(y''(t)) = s^2 l(y(t)) - sy(0) - y'(0)$ or
  $l(ty''(t)) = -2sl(y(t)) - s^2\frac{d}{ds}l(y(t)) + y(0)$
- Laplace transform of integral: $l\left(\int_0^t f(r)dr\right) = \frac{F(s)}{s}$
- Convolution of functions: $(f * g)(t) = \int_0^t f(r)g(t-r)dr$
- Integral equation with integral of the form of convolution:
  $y' + \int_0^t y(r)f(t-r)dr = f(t)$
  $\Leftrightarrow y' + y(t) * f(t) = f(t)$

...cont'

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